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A CORRECTION AND SOME ADDITIONS TO "REPARAMETRIZATION OF n-FLOWS OF ZERO ENTROPY"

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ABSTRACT. In addition to correcting an error in the previously mentioned paper, we show that if $v \mapsto \varphi_w$ and $w \mapsto \psi_\sigma$ on X and Y are n- and m-flows, respectively, then the $(n + m)$-flow $(v, w) \mapsto \varphi_v \times \psi_w$ on $X \times Y$ is "loosely Kronecker" if and only if φ and ψ are.

There is a silly and easily correctable mistake in our paper **[4]**. Recall that an n-flow is a free, ergodic, probability-preserving action of $\mathbf{R}^n$. We constructed in **[4]** an action φ of $\mathbf{R}^n$ as follows: $t \mapsto T_t$ was defined as a suspension over the non-LB, ergodic, zero-entropy transformation of **[2]**. Then, for an $(n - 1)$-vector u, $\varphi_{(t,v)}$ was defined as T_t. φ is indeed ergodic and probability-preserving, but it is not free, so of course it is not an n-flow.

The purpose of the construction was to produce a zero-entropy n-flow which is not LK in the sense of **[4]**. First, we would like to change terminology, and use the term "standard" (as in Katok **[5]**) rather than "LK". The object, then, is to construct a nonstandard n-flow of zero entropy. One way would be to fix up the prior example as follows: let the above flow T act on (Y, ν), and let θ be any $(n - 1)$-flow on a space (Z, ρ). Then $\varphi_{(t,u)} = T_t \times \theta_u$ will be a nonstandard n-flow of zero entropy. That it is nonstandard may be seen as in the argument given at the end of **[4]** and it is easy to see that it has zero entropy. However, we now give a sketch of a more enlightening approach to the matter.

First, we point out

LEMMA 1. *A standard n-flow has entropy zero.*

The easiest way to see this is to use the ideas of r-entropy, from **[3]**: to say φ is standard is to say that for large N, most C_N names for $(\varphi, \mathcal{P})$ are f_N-close. The Lebesgue continuity theorem then may be used to get an exponentially small bound on the number of sets of d_N diameter r which are required to cover most of the space on which φ acts. $\square$

Hereafter, let ψ be an l-flow on (Y, ν) and θ an m-flow on (Z, ρ). If $\varphi_{(t,u)} = \psi_t \times \theta_u$, then φ is an $(l + m)$-flow on $(Y \times Z, \nu \times \rho)$.

LEMMA 2. *φ as above necessarily has entropy zero.*

INDICATION OF PROOF. By using partitions of the form $\mathcal{R} \times \mathcal{S}$ where $h(\psi, \mathcal{R})$ and $h(\theta, \mathcal{S})$ are finite, we may reduce to the case where ψ and θ have finite

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entropy. But now the result follows directly from the definition of entropy, essentially because $(l+m)N/N^{l+m} \to 0$ as $N \to \infty$. See **[1]** for discussions of this type in the discrete case.

LEMMA 3. *φ as above is standard if and only if both ψ and θ are.*

PROOF. If both ψ and θ are standard, then for any partition of the form $\mathcal{R} \times \mathcal{S}$, the process $(\psi \times \theta, \mathcal{R} \times \mathcal{S})$ may be seen to be standard by doing f-matching for $(\psi, \mathcal{R})$ and $(\theta, \mathcal{S})$ separately, and then combining.

To go in the other direction, one may use a similar argument to that at the end of **[4]** to make a reduction of dimension. Here are the details.

Suppose φ is standard. Choose a partition $\mathcal{P}$ of Y. Then $\mathcal{R} = \{P \times Z : P \in \mathcal{P}\}$ is a partition of $Y \times Z$. So, referring to the definitions in §3 of **[4]**, we see that for any $\varepsilon > 0$ there is some $M > 0$ such that if $M < N$ there is a set $E_N \subset Y \times Z$ with $\nu \times \rho(E_N) > 1 - \varepsilon$ and $f_N^{\mathcal{R}}(x, x') < \varepsilon$ whenever $x, x' \in E_N$. There is thus some $z \in Z$ so that if we set $F_N = \{y: (y, z) \in E_N\}$ then $\nu(F_N) > 1 - \varepsilon$. Now, if $t \in R^l$ and $u \in R^m$, then $\mathcal{R}(y, z)(t, u) = \mathcal{P}(y)(t)$, independent of z. So $f_N^{\mathcal{R}}((y, t), (y', t')) < \varepsilon$ provided $y, y' \in F_N$. So for any such y, y', and any z, z', there is some $h \in \mathfrak{D}_{C_N^{l+m}}$ such that

$$\frac{1}{|C_n^{l+m}|} \int_{C_n^{l+m}} \delta(\mathcal{R}(y, z)(h(t, u)), \mathcal{R}(y', z')(t, u))\, dt\, du < \varepsilon.$$

(Since there are different dimensions to worry about, we now denote the N-cube in R^p by C_N^p.) Rewriting, and writing $h(t, u)$ as $(j(t, u), k(t, u))$, where $j : R^{l+m} \to R^l$ and $k : R^{l+m} \to R^m$, we have

$$\frac{1}{|C_N^l|} \frac{1}{|C_N^m|} \int_{C_N^l} \int_{C_N^m} \delta(\mathcal{P}(y)(j(t, u)), \mathcal{R}(y')(t))\, dt\, du < \varepsilon;$$

so for some u_0 we have

$$\frac{1}{|C_N^l|} \int_{C_N^l} \delta(\mathcal{P}(y)(j(t, u_0)), \mathcal{R}(y')(t))\, dt < \varepsilon.$$

Set $i(t) = j(t, u_0)$. i is a differentiable function from C_N^l to C_N^l leaving fixed a neighborhood of the boundary. Furthermore $\|i' - I_{R^l}\|_\infty \leqslant \|h' - I_{R^{l+m}}\|_\infty < \varepsilon$. Finally, assuming $\varepsilon < 1$, we have $\|i'(y) - I_{R^l}\| < 1$ for each y, so i is locally invertible (by the Inverse Function Theorem), so–since C_N^l is simply connected–i is globally invertible, i.e. $i \in \mathfrak{D}_{C_N^l}$. Thus $f_N^{\mathcal{P}}(y, y') < 2\varepsilon$ for all $y, y' \in F_N$. But ε was arbitrary, so we are done. □

It is now easy to produce, for $n \geqslant 2$, examples of nonstandard n-flows of zero entropy: just take ψ to be a 1-flow of positive entropy, and θ any $(n-1)$-flow whatsoever. Then by Lemma 2, φ will have zero entropy. It cannot be standard, because if it were, then by Lemma 3, ψ would also be standard, and therefore by Lemma 1 would have to have entropy zero.

Alternatively, one could, as in **[4]**, take ψ to be some nonstandard 1-flow of zero entropy. Such examples are provided by proving the following fact:

LEMMA 4. *A flow is standard in the present sense if and only if it is LB in the sense of* **[2]** *and of zero entropy, or, equivalently, standard in the sense of* **[5]**.

The proof is a fairly routine application of the definitions. This construction is in principle more difficult, in that it already needs the existence of non-LB flows in one dimension. However, it may be useful in constructing uncountably many different equivalence classes.

The first construction, setting $\varphi_{(t,u)} = \psi_t \times \theta_u$ with ψ of positive entropy, raises the interesting possibility of exhibiting some "natural" equivalence classes other than the standard class, among the entropy zero n-flows, $n \geqslant 2$.

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